Closed Form Solution for Maximizing the Sum Capacity of Reverse-Link CDMA System with Rate Constraints

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Abstract— In this paper, we work on maximizing the capacity of the reverse link of a CDMA wireless network. Oh and Soong [1] presented an information–theoretic capacity model for analyzing this system. That formulation produces solutions that often result in unfairness allocation of power to most users. Here, we propose the addition of a maximum capacity constraint into the problem, thereby creating a more fair power allocation scheme. First, we improve on the algorithm presented in [1] by proposing a closed form solution and an algorithm which works nine times faster. Then, we propose a closed form algorithm for solving the new enhanced problem.

Index Terms— Quality of service, single cell, reverse link CDMA, optimization.

I. INTRODUCTION

WITH the introduction of multimedia services to wireless CDMA communications, the goal is no more to provide fixed capacity to all users [2], but to maximize the aggregate capacity given a set of constraints [3]. In this paper, we focus on maximizing the capacity of the reverse link (uplink), which is often the limiting link [4]. Note that we are looking at the traffic channels, and not the access channels [5]. We emphasize that power control strategies used in the reverse link are completely different from those utilized in the forward link [6]. This is mainly due to the stringent requirements of the reverse link [7].

Maximizing the capacity of the reverse link in this paper is achieved through controlling the transmission power of the individual stations. The analysis is carried out in a single–cell system by assuming that either there is only one cell in the system or that the activity in other cells can be modeled as interference to the current cell [8]. We also use the term capacity as the rate of transmission of each station. To relate transmission power to rate, we use Shannon's theorem as a maximum bound [3], [9]. This approach is adopted from previously developed models (see [10] for example).

The system–wide information theoretic capacity of CDMA systems was first analyzed in [8] and then developed further for multi–user networks [11], [3]. While these works focused on the set of all capacities, more recent research has benefited from the advances in matched filters and has analyzed the aggregate capacity [10], [12], [13]. In these works, assuming a sub–optimal coding scheme, the mapping between signal to interference ratio (SIR) and throughput is determined by the coding strategy. A next development has been the assumption of Shannon capacity. Although, Shannon only gives the maximum bound for the capacity, the existence of coding strategies such as Turbo Coding makes the Shannon bound practically reachable [3].

Here, we first formally define the problem. Assume that there are M mobile stations with reverse link gains of $g_1, \cdots, g_M$, which satisfy $g_1 > \cdots > g_M$. Denote the transmit power of the i–th mobile station as p_i, $0 \leq p_i \leq p_{max}$ where p_{max} is the maximum transmission power of each station. With a background noise of I, the signal to noise ratio (SNR) for the signal coming from the i–station, as perceived by the base station, is calculated as,

$$\gamma_i = \frac{p_i g_i}{I + \sum_{j=1, j \neq i}^{M} p_j g_j}. \tag{1}$$

Using $C = B \log_2 (1 + \gamma)$ and omitting the constant B, the aggregate capacity of the system can be written as,

$$C(\vec{p}) = \log_2 \frac{\left(I + \sum_{j=1}^{M} p_j g_j \right)^M}{\prod_{i=1}^{M} \left(I + \sum_{j=1, j \neq i}^{M} p_j g_j \right)}. \tag{2}$$

The aim of the optimization problem formulated in [1], which has been improved here, is to find the vector $\vec{p} = (p_1, \cdots, p_M)$ which maximizes $C(\vec{p})$ subject to these constraints,

$$\begin{cases} \forall i, 0 \leq p_i \leq p_{max} \\ \forall i, \dfrac{p_i g_i}{I + \sum_{j=1, j \neq i}^{M} p_j g_j} \geq \gamma \\ \sum_{i=1}^{M} p_i g_i \leq P_{max} \end{cases}. \tag{3}$$

We call this the classical problem. In [1], the authors showed that the search space of this problem can be reduced from $[0, p_{max}]^M$ to a finite number of one–dimensional spaces. Then, to locate the optimal point, they used a numerical search. This solution was observed to lead to one user receiving a disproportionately higher capacity (power allocation) than the remaining users. This type of result we say is unfair to some users. In addition we find that the algorithm developed in that paper could be improved significantly.

In this paper, we go further and prove that the search space can be reduced to an order of M points, compared to the order of M intervals given in [1]. This way, we eliminate the numerical search stage used in [1] and achieve a higher level of performance and accuracy. We show that using these results the problem is solved much faster. Then, to solve the known unfairness issue about the problem, we add a maximum capacity constraint. Subsequently, we propose a closed form

algorithm for solving the new problem and show that the unfairness is limited in the new problem formulation. Finally, we compare the two problems in a time interval, for moving stations.

II. CLASSICAL PROBLEM: IMPROVEMENT TO ITS SOLUTION TECHNIQUE

In this section we present the summary of our improvements to the classical problem.

Defining the variable $x_i = p_i g_i / I$, the classical problem can be rewritten as minimizing,

$$\Phi(\vec{x}) = \frac{\prod_{i=1}^{M} \left(1 + \sum_{j=1}^{M} x_j - x_i\right)}{\left(1 + \sum_{j=1}^{M} x_j\right)^M}, \tag{4}$$

subject to,

$$\begin{cases} \forall i, 0 \leq x_i \leq l_i = \dfrac{p_{max}}{I} g_i \\ \forall i, \dfrac{x_i}{1 + \sum_{j=1}^{M} x_j} \geq \varphi = \dfrac{\gamma}{\gamma + 1} \\ \sum_{i=1}^{M} x_i \leq X_{max} = \dfrac{P_{max}}{I} \end{cases} \tag{5}$$

Because of the special structure of (4) and (5), we focus on the hyperplanes defined as $\sum_{i=1}^{M} x_i = T$ for fixed values of T, independently. In such a hyperplane, the problem can be restated as finding the minimum of $\Phi(\vec{x}) = (1+T)^{-M} \Phi_T(\vec{x})$ where $\Phi_T(\vec{x}) = \prod_{i=1}^{M} (1 + T - x_i)$. Then, having fixed T, the problem is to minimize $\Phi_T(\vec{x})$. Furthermore, in these hyperplanes, the constraints can be written as $T \leq X_{max}$ and $\forall i, \varphi(1 + T) \leq x_i \leq l_i$.

Theorem I: Consider any $i \neq j$, and δ, changing x_i and x_j into $x_i + \delta$ and $x_j - \delta$, respectively, gives another point on the same hyperplane. In the solution to the classical problem, we establish that $|(x_i + \delta) - (x_j - \delta)| \leq |x_i - x_j|$.

For proof of this theorem, as well as the proofs for other theorems given in this paper, refer to [14], [15]. Then, a modified version of Proposition 1 in [1] is stated as Theorem II, which gives the structure of the optimum solution to the classical problem. Here, the unknowns are k, x_k, and T..

Theorem II: If there is any solution to the classical problem, then it is structured as $\vec{x} = [l_1, \cdots, l_{k-1}, x_k, \varphi(1 + T), \cdots, \varphi(1 + T)]$, for a value of k.

The work introduced in [1] stops at this point and uses numerical search for different values of k to find the best x_k. Here, we go further and reduce the size of the search space by finding appropriate necessary and sufficient bounds for both k and x_k, using Theorem III. This theorem also shows how T is related to k and x_k.

Theorem III: In the optimal solution to the classical problem, we have $T = \frac{x_k + L + 1}{1 - (M-k)\varphi} - 1$, where $L = \sum_{i=1}^{k-1} l_i$. Furthermore, k must satisfy $M - \frac{1}{\varphi} + 1 < k < \frac{1}{\varphi} - \frac{1}{l_M} + 1$ and x_k must comply with $min_k \leq x_k \leq max_k$ for (6). All these inequalities are both necessary and sufficient.

By reducing the search space from a set of one–dimensional spaces into a finite set of points we present in Theorem IV how to determine the optimal x_k, for a known k. This is another

important contribution of this paper, compared to [1], where the plausible interval for x_k is numerically searched.

Theorem IV: In the optimal solution to the classical problem, x_k must accept one of the marginal values given in Theorem III.

Using the results produced so far we propose the algorithm Classical Single–Cell (CSC) (for details refer to [14]). This algorithm searches for the optimal solution among less than $2M$ points, for which closed forms are given. Analysis of the proposed algorithm shows that its computational cost is $O(M^2)$ flops, $8M^2$ flops in the worst case and $2M^2$ flops in a typical one. The old algorithm needs $75M^2$ flops. Comparing this with the computational complexity of the proposed algorithm, the proposed one is at least more than nine times faster, usually more than thirty six times faster. It is worth to mention that it can be shown that the actual value of P_{max} is important only if $P_{max} < p_{max} g_M \frac{\gamma+1}{\gamma} - I$.

III. WEAKNESS OF THE OLD PROBLEM

Experimental analysis shows that in the solution to the CSC problem, every station, except for the first one, is set to transmit at the minimum possible SNR. The first station's transmit power, on the other hand, is only limited by the maximum aggregate power received, or its own maximum power limit. This results in a ratio unfairness of over a thousand, in most cases, where the ratio unfairness, $\tilde{f}$, is defined as the ratio between the maximum and the minimum capacities of the stations.

This was observed in [1], [12], [13]. In fact, this unfairness was one of the main reasons the authors in [1] included the minimum SNR constraint into the problem. We conjecture that the problem with the solution comes from the fact that the constraints neglect the unfairness of the solution. Furthermore, the formulation ignores the fact that a maximum bound on the capacities of the single stations is also necessary.

IV. NEW PROBLEM

Based on the discussion given in Section III, we propose a new constraint to be added to the optimization problem, as $\forall i, C_i \leq \eta$. Now, using the definition of x_i, and by defining $\omega = 1 - 2^{-\eta}$, the problem reduces to minimizing $\Phi_T(\vec{x})$ given $T \leq X_{max}$ and $\forall i, \varphi(1 + T) \leq x_i \leq \min\{l_i, \omega(1 + T)\}$. Note that, because the objective function of this problem is identical to the one for the classical case, Theorem I is still valid. Furthermore, looking at the search space of the new problem, it is a pyramid–like structure. While in the classical problem the maximum bounds were decreasing, in the new problem they are fixed for some initial values of i, where $l_i > \omega(1 + T)$, and then they start decreasing. Thus, the three lemmas given in [1] are still valid and a theorem similar to Theorem II can be proved similarly for the new problem.

Theorem V: If there is any solution to the new problem, then it is structured as $\vec{x} = [\omega(1 + T), \cdots, \omega(1 + T), l_{j+1}, \cdots, l_{k-1}, x_k, \varphi(1 + T), \cdots, \varphi(1 + T)]$, for a pair of j and k.

Theorem VI: In the optimal solution to the new problem we have $T = \frac{1}{\psi}(x_k + L + 1) - 1$ where $L = \sum_{i=j+1}^{k-1} l_i$ and $\psi = 1 - [j\omega + (M - k)\varphi]$. Furthermore, we must have $\psi > \varphi$ and $min_{jk} \leq x_k \leq max_{jk}$ for,

$$\begin{cases} max_k = \min\left\{ l_k, (X_{max}+1)\left[1-(M-k)\varphi\right]-L-1, l_M\dfrac{1-(M-k)\varphi}{\varphi}-L-1 \right\} \\ min_k = \dfrac{\varphi(L+1)}{1-(M-k+1)\varphi} \end{cases} \qquad (6)$$

Station #		1	2	3	4	5	6	7	8	9	10
$g_i\ (\times 10^{-12})$		0.52	0.018	0.016	0.0091	0.0082	0.0081	0.0075	0.0059	0.0059	0.0045
	P	b	x	x	x	x	x	x	x	x	x
	p_i	46.5266	5.2114	5.8548	10.4390	11.6260	11.7556	12.6169	16.0532	16.1979	20.9157
Classical	C_i	2.3606	0.0046	0.0046	0.0046	0.0046	0.0046	0.0046	0.0046	0.0046	0.0046
Problem	$\tilde{C}_i$	98.2%	0.2%	0.2%	0.2%	0.2%	0.2%	0.2%	0.2%	0.2%	0.2%
	C					2.402					
	f					2.356					
	$\tilde{f}$					518.2					
	P	X	l	l	l	l	l	l	l	l	b
	p_i	14.662	16.092	16.189	16.584	28.796	36.313	50.000	13.744	14.535	18.984
New	C_i	0.3000	0.2111	0.1863	0.1015	0.0908	0.0898	0.0835	0.0652	0.0646	0.0498
Problem	$\tilde{C}_i$	24.1%	17.0%	15.0%	8.2%	7.3%	7.2%	6.7%	5.2%	5.2%	4.0%
	C					1.243					
	f					0.2502					
	$\tilde{f}$					6.027					

$$\begin{cases} max_{jk} = \min\begin{Bmatrix} l_k, \\ \dfrac{\omega}{\psi-\omega}(L+1), & \psi>\omega, \\ \dfrac{\psi}{\omega}l_j-(L+1), & j\neq 0, \\ \dfrac{\psi}{\varphi}l_M-(L+1), \\ \psi(X_{max}+1)-(L+1), \end{Bmatrix} \\ min_{jk} = \max\begin{Bmatrix} \dfrac{\psi}{\omega}l_{j+1}-(L+1), & k>j+1, \\ \dfrac{\varphi}{\psi-\varphi}(L+1) \end{Bmatrix} \end{cases} \qquad (7)$$

Theorem VII: In the optimal solution to the new problem, x_k must accept one of the marginal values given in Theorem VI.

Using Theorem VII we propose the algorthm New Single Cell (NSC) (for details refer to [15]). This algorithm searches for the optimal solution among less than $2M^2$ points, for which closed forms are given. Analysis shows that the computational cost of this algorithm is $O(M^3)$, $5.3M^3$ flops in the worst case and $1.3M^3$ flops in a typical one. We define the subtractive unfairness f as the difference between these two entities.

It can be shown that in the new problem,

$$f \leq \log_2\left(\frac{1-\varphi}{1-\omega}\right) \simeq \eta - \frac{1}{\ln 2}\gamma, \tilde{f} \leq \frac{\log_2(1-\omega)}{\log_2(1-\varphi)} \simeq \frac{\eta}{\gamma}\ln 2. \quad (8)$$

Note that such bounds on the unfairness of the system did not exist in the classical problem.

V. EXPERIMENTAL RESULTS

The computer codes for the proposed algorithms are developed in MATLAB 7.0.4 on a PIV 3.00GHZ with 1GB of RAM. The parameters used in this study are $\gamma = -50dB$, $I = -113dBm$, and $P_{max} = -106dBm$, $p_{max} = 23dBm$ [16],

[17] plus $\eta = 0.3$. Here, we work on a circular cell of radius $R = 2.5Km$. For the station i at distance d_i from the base station, we only consider the path loss which is calculated as $g_i = Cd_i^n$ [18]. Here, C and n are constants equal to 7.75×10^{-3} and -3.66, respectively, when d_i is in meters [16].

To compare the results of the two formulations, the same problems are solved in both frameworks. Table I shows the values of g_i for a sample example where M is equal to 10. Table I also shows the solutions to the classical and new problems with the set of parameters given in the above. For each problem, first the pattern of the solution is given. Using this pattern we are interested in seeing where the breaking points occur. As also expected from previous research [12], [13], the classical solution serves the first station with the maximum possible capacity while the others are left to the minimum guaranteed amount. When we add the maximum capacity constraint, we are in fact limiting the first station's capacity. As seen in the results of the new problem, this results in a more balanced spread of the capacity between more users. Looking at the solution to the new problem, nine users are served at the maximum possible capacity (determined either by η or l_i) while the tenth one has a value in between.

To show this difference more clearly, we define the capacity share of the i–th station as $\tilde{C}_i = \frac{C_i}{C}$. The capacity shares in both cases are shown in Table I. It is clear that the classical problem produces a solution which is dominantly dependent on the first station, while the solution to the new problem spreads the capacity more evenly among more number of stations.

As expected both from the analysis and from the set of capacity values shown here, the solution to the new problem is more fair than the one produced by the classical problem. In
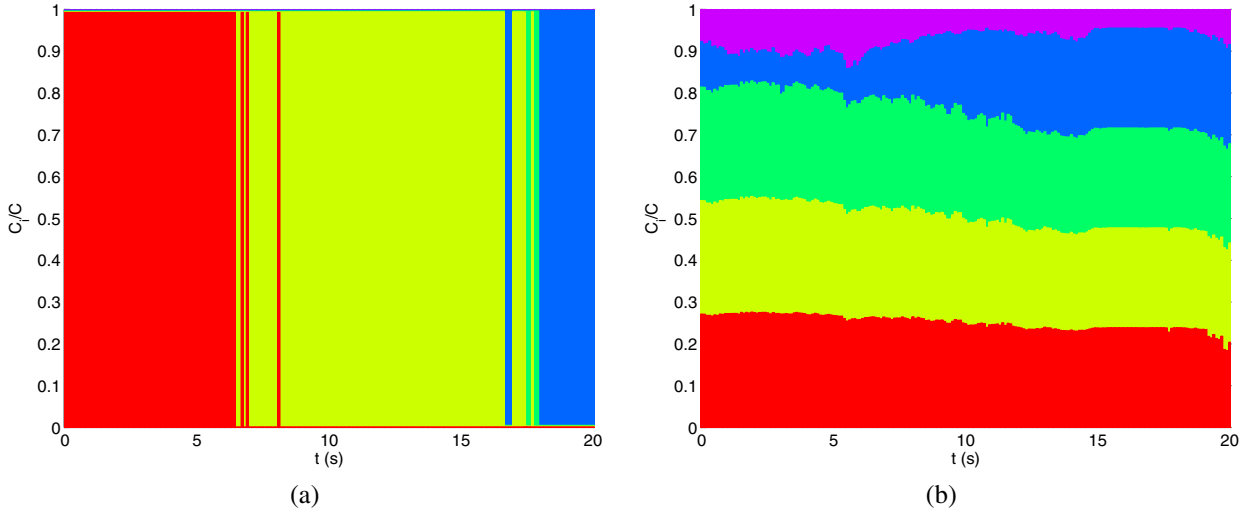
Fig. 1. Capacity shares of different stations over time shown by different colors. (a) Classical problem. (b) New problem.
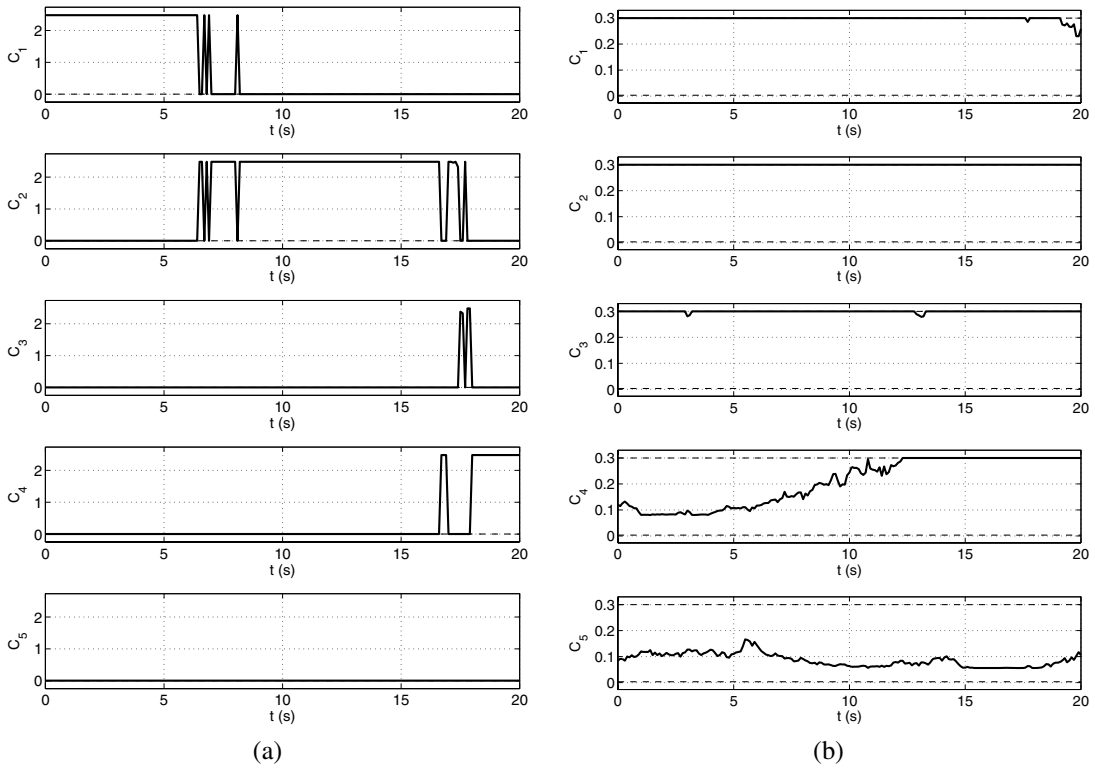

Fig. 2. Capacity of different stations over time. (a) Classical problem. (b) New problem.

fact, the ratio unfairness measure is about eighty times lower in the new problem compared to the classical one; 6 compared to 518. Note that, here, we see the aggregate capacity becoming smaller in the new problem as a result of adding the maximum capacity constraint.

Here, we carry out another experiment in which $M = 5$ stations move around in a cell as modeled by a discrete random walk with the speed at each moment chosen as a uniform random variable between zero and $5Km/h$ [19]. Here, we assume that no station leaves or enters the cell. Now, looking at a time span of $20s$, the system is solved every $100ms$ using both algorithms, independently. It is worth to mention that

CPU utilization is 7% and 36% for the classical problem and the new one, respectively.

Figure 1 shows the capacity shares of different stations in the two problems over time, where each color indicates one station. As anticipated, the classical problem tends to allocate its entire resources to the closest station at each moment. On the contrary, the new problem only increases its dependency upon each station as they get closer to the base station. Comparison also shows that the classical problem tends more to force stations to have rapid changes in the power [15]. Furthermore, the capacity curves for the classical problem, given in Figure 2–a, show that the stations are mostly

oscillating between two situations of minimum capacity and a very high one. On the other hand, the new problem (see Figure 2–b) keeps the capacity of different cells between the two specified bounds. The significant observation about the solution to the new problem in this example is that the capacity of none of the stations is ever squeezed down to the minimum guaranteed limit (this may not always be true).

VI. CONCLUSIONS

In this paper, the problem of optimizing the aggregate system capacity of the reverse link in a single–cell CDMA system was addressed. The formulation was adopted from previous research which also suggested a numerical algorithm for solving the problem. Here, we proposed a closed form algorithm which solves the problem at least nine times faster. Then, to solve the unfairness issue about the problem, we proposed a maximum capacity constraint to be added to it and proposed another closed form algorithm for solving the resulting problem. We also analytically showed that the new constraint gives an upper limit for the unfairness of the solution. This was also presented by solving the two problems for the same set of stations and parameters. Then, we adopted a scenario in which a few stations move around in a cell. We showed that the classical problem tends to limit all stations' capacities to the minimum guaranteed values, while the closest station is granted a very high capacity. The new problem, on the other hand, only increases its dependency on the stations, as they become closer to the base station.

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